GENERALIZATION IN FIFTH GRADERS WITHIN A FUNCTIONAL APPROACH

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This article discusses evidence of 24 fifth graders' (10-11 year olds') ability to generalize when solving a problem which involves a linear function. Analyzed in the context of the functional approach of early algebra, the findings show that three students generalized both when solving specific instances and when asked to provide the general formula; while 16 students generalized only when asked to define the general formula. The results are described in terms of the functional relationship identified, the types of representation used to express them, and the type of questions in which students generalized their answers. Most of the pupils who generalized did so based on the correspondence between pairs of values in the function at issue.

Keywords: Functional relationship; Functional thinking; Generalization; Representations

Generalización de estudiantes de quinto de primaria desde un enfoque funcional

En este artículo presentamos evidencias de generalización de 24 estudiantes de quinto de primaria (10-11 años) al resolver un problema que involucra una función lineal. Desde el enfoque funcional del early algebra*, los hallazgos muestran que tres estudiantes generalizaron al trabajar con casos particulares y cuando se les pidió expresar la regla general; mientras que 16 estudiantes solo lo hicieron cuando les pedimos expresar la regla general. Describimos los resultados en términos de la relación funcional identificada, los tipos de representaciones que emplearon para expresar dichas relaciones y el tipo de pregunta en la cual los estudiantes generalizaron. La mayoría de los estudiantes que generalizaron establecieron una relación de correspondencia entre los pares de valores de la función.*

Términos clave: Generalización; Pensamiento functional; Relaciones funcionales; Representaciones

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Research interest in mathematics education is growing around elementary school students' understanding and expression of notions about algebraic concepts, particularly these students' generalization and how they express general relationships when solving different problems (Carpenter & Levi, 2000; Kaput, 2008). In this context, algebraic thinking plays a key role in research on school algebra, for it entails the development of the ability to analyze relationships between quantities, identify general patterns and use symbols to represent ideas, among others (Kaput, 2008; Kieran, 2004). Different researchers denote that "developing children's ability to generalize in the elementary grades is vital because it draws their attention away from the particulars of arithmetic instances and onto the relationships and structure that connect these particular instances" (Blanton, Brizuela, & Stephens, 2016, p. 2).

Generalization is one of the main components of algebraic thinking and, particularly, of functional thinking, which is the focus of this study. This type of thinking addresses in the function and the relationship between two (or more) variables: specifically, it involves the types of thinking that range from specific relationships to the generalization of relationships (Smith, 2008). Although such functional thinking appears to be beneficial for students, its application in the elementary grades has received scant attention (Blanton & Kaput, 2011).

Generalization and representation notions are very close in Elementary education (Kaput, Blanton, & Moreno, 2008). Traditionally, generalization process has been linked to the use of algebraic symbolism and much of the previous research has focused on the students' difficulties in generalizing (Dienes, 1961; Ellis, 2007). For this reason, it was not clear if difficulties arose from the generalization process or from the representation used. Students at Elementary, or even at Kindergarten, are able to identify regularities that serve as first approach to generalization with no knowledge about algebraic symbolism (e.g., Castro, Cañadas, & Molina, 2017; Schifter, Monk, Russell, & Bastable, 2008).

Our focus is on describing fifth graders' generalization when solving tasks which involve functions. Specifically, our interest is to get information about how and when these students generalize. From the research literature review, we conjecture that students: (a) establish different relationships between the variables involved, and (b) generalize using different representations.

Theoretical Framework

In this section, we describe some ideas concerning our research problem.

Generalization

According to some researchers, generalization is the key element in algebra (e.g., Mason, 1996). It is present when students intuitively perceive a certain underlying pattern, even though they are unable to represent it clearly (Mason, Burton, & Stacey, 1988). Generalization is an adaptation and reorganization process in which a person identifies the essence of one idea, which implies deliberate reasoning that builds on specific cases to identify inter-model, inter-procedural or inter-structural relationships (Carraher & Schliemann, 2002; Kaput, 1999; Mitchelmore & White, 2007).

Different researchers show types or levels of generalization (e.g., Harel & Tall, 1991; Kaput, 1999; Kruteskii, 1976; Stacey, 1989). In the functional approach of early algebra, Blanton, Brizuela, Gardiner, Sawrey and Newman-Owens (2015) show different ways in which students relate variables focused on functional relationships. Their findings distinguished between students who identified a specific and those who detected a general relationship between variables, and related the distinction to the ability to symbolize. Students who established the relationship between variables for specific cases "did not yet have a representational means to compress multiple instances into a unitary form that could symbolize these instances" (p. 542). Pinto and Cañadas (2017) describe fifth graders' generalization when solving different items from a written questionnaire, identifying: (a) spontaneous generalization, when students generalize when answering questions about particular cases; and (b) prompted generalization, when students generalized when answering a question involving the general case. Carraher, Martinez and Schliemann (2008) establish different criteria to describe students' generalization in their last years of elementary education. These criteria are: (a) form of the underlying function, (b) variables mentioned, (c) types of operations used, (d) use (or not) of algebraic notation, (e) structural features, and (f) the meaning of the different components of the written formula.

Algebraic symbolism has been directly associated with generalization in different grades. In the context of elementary grades, some authors highlight the students' use of different representations to express a general relationship. For example, Mason and Pimm (1984) describe the use of everyday language as a fundamental resource to express generalization and its use can have influence in the use of algebraic symbolism to express the generalization. Radford (2010) recognizes the importance of gestures as a way to express generalization. Moreover, other types of representation, including verbal, numerical, pictorial and manipulative, are of interest in the context of early algebra (Blanton, Levi, Crites, & Dougherty, 2011; Merino, Cañadas, & Molina, 2013). In summary, and with base on previous ideas, we assume that the generalization from a functional approach at elementary grade can be expressed in different representations.

Functional Thinking

Functional thinking is a "component of algebraic thinking based on construction, description and reasoning with and about functions and their constituents" (Cañadas & Molina, 2016, p. 210) that ranges from specific relationships to generalizing the relationships between two (or more) variables (Smith, 2008). In most countries, students are not introduced to functions, which comprise the core content of this type of thinking, until secondary school.

The present study used the linear function $f(x)=ax+b$ (with the domain and codomain limited to natural numbers) as a port of entry for early algebra to afford students the opportunity to: (a) explore variations in quantities, (b) use different representations, (c) expand numerical contents working with domain and codomain of variables, (d) help to increase students' ability to get generalization, among others (Blanton et al., 2011; Carraher & Schliemann, 2018; Romberg, Fennema, & Carpenter, 1993).

Our work is focused on bivariate functions. Smith (2008) defined the functional relationships involving two quantities that co-vary to be: (a) correspondence, or the relationship between the pairs of values for the two variables $(a, f(a))$; and (b) covariation, or the relationship that describes how changes in one variable affect the other. These relationships can be express through different representations and can relate particular values of the variables or the general case. Some authors describe that whenever students work with particular co-varying quantities, their ability to identify the functional relationship increases (Warren, Miller, & Cooper, 2007).

METHOD

This study forms part of a broader teaching experiment on functional thinking in fifth graders in which the contextualized problem posed in each session revolved around a linear function. This article discusses the results of the fourth and final session, when student progress was greatest because they had already worked on a number of problems involving functions.

Subjects and Tools

The 24 subjects were fifth graders (10 to 11 year-olds) enrolled in a school in the South of Spain, who were deliberately chosen on the grounds of school and teacher availability. The students had not worked on problems involving functions prior to the study, except in the first three sessions of the teaching experiment, in which they worked with problems that involve linear functions. In the Table 1 we present a summary of the first three sessions.

Table 1
Contexts and Functions in Each Session

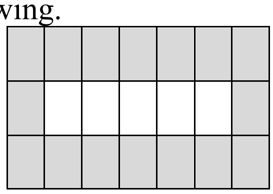
Session	Context	Function
1	Carlos wants to sell shirts for earning money for a study trip with his class. He earns 3 euros per shirt.	$f(x)=3x$
2	Daniel and Laura sell different shirts for their study trip. Laura gets 3 euros for each shirt. Daniel has saved 15 euros. For each shirt he gets 2 euros.	$f(x)=3x$ and $f(x)=2x+15$

Table 1
Contexts and Functions in Each Session

Session	Context	Function
3	Juan has saved some money (he only has euros, not cents). His grandmother wants to reward him for a job he has done for her. She offers him two deals: Deal 1. I'll double the money you have Deal 2. I'll give you triple your money and you give me 7 euros.	$f(x)=2x$ and $f(x)=3x-7$

The research team consisted in the teacher-researcher who led the sessions and two researchers who recorded the videos and helped answer students' questions. In the tiles problem posed to all students, the implied function was $f(x)=2x+6$. The problem and related questions are reproduced in Figure 1.

A school wants to re-pave its corridors because they are in poor condition. The school decides to use a combination of white and grey tiles, all square and all the same size. They are to be laid as in the drawing.



The school contracts a company to re-pave the corridors on all three floors. We want you to help the workers answer some questions before they get started.

Q1. How many grey tiles will they need for a corridor with 5 white tiles?

Q2. Some corridors are longer than others. So the workers will need a different number of tiles for each corridor. How many grey tiles will they need for a corridor with 8 white tiles?

Q3. How many grey tiles will they need for a corridor with 10 white tiles?

Q4. How many grey tiles will they need for a corridor with 100 white tiles?

Q5. The workers always lay the white tiles first and then the grey tiles. How can they figure out how many grey tiles they need if they have already laid the white ones?

Figure 1. The tiles problem

The questions posed involve: (a) specific instances (Q1, Q2, Q3 and Q4), and (b) the general case (Q5).

The information gathered included the session videos and the students' answers to the questionnaire. This article describes the results from the students' written responses.

Analytical Categories and Data Analysis

The theoretical framework and background were applied to define some of the categories. Generalization was identified based on its presence or absence in students' replies to questions Q1 to Q5. In the Table 2 we present the categories with which we describe the generalization of students.

Table 2
Analytical Categories

Categories	Description
Type of generalization	We distinguished between the prompted generalization (generalization to Q5) or spontaneous generalization (generalization to Q1-Q4) (Pinto & Cañadas, 2017).
Functional relationship identified	Students' generalization was described in terms of the functional relationship generalized: correspondence or covariation (Smith, 2008).
Representations	Representations used by the students to generalize: pictorial, verbal, numerical or with algebraic notation, as well as combinations of one or the other or both with other representations (Blanton et al., 2011; Carraher et al., 2008).

Students were labeled as Si, where i = 1, ..., 24.

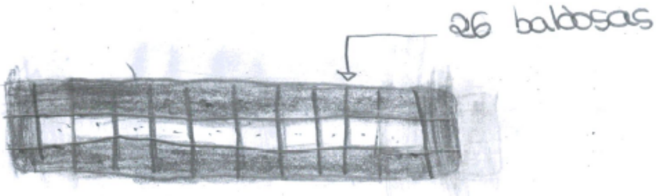
RESULTS

We present a first approach to the results in Table 3, distinguishing those students who did not generalize and those did, attending to the categories previously described.

Table 3
Evidence of Students' Generalization

Non generalization	Generalization		
	Only prompted	Only spontaneous	Both
S15, S16, S18, S19, and S20	S1, S2, S3, S4, S7, S9, S10, S11, S12, S13, S14, S17, S21, S22, S23, and S24		S5, S6, and S8

Table 3 illustrates that of the 24 students, five students did not generalize in their responses to Q1-Q5. These students gave direct answers only (i.e., only the numerical result), described how they counted the tiles, repeated the problem, or made a drawing: no generalization could be attributed to these pupils. For instance, we present the S19's answer to Q4 in Figure 2. He made a drawing to represent the number of grey tiles needed for 10 white tiles.



Note. Baldosas means tiles in Spanish.

Figure 2. Non-generalization's example, S19 in Q4

On the other hand, 19 students generalized the functional rule and two profiles were identified: (a) three students exhibited both spontaneously and prompted generalization, and (b) 16 students generalized promptly (when replying to Q5). A discussion of these types of responses follows.

Spontaneous and Prompted Generalizations

Three students exhibited both spontaneously and prompted generalization. We present in Table 4 questions where these students evidenced generalization and the representation used.

Table 4
Spontaneous and Prompted Generalization and Representation Used

Student	Question 1	Question 2	Question 3
S5	✓ Algebraic		✓ Verbal
S6	✓ Verbal	✓ Verbal	✓ Algebraic
S8	✓ Algebraic		✓ Verbal

As we can observed in Table 4, three students (S5, S6, and S8) generalized in questions 1, 2, and 5. Two of them generalized (S5 and S8) in Q1 and Q5, and S6 in Q1, Q2 and Q5. These three students used verbal and algebraic representations to generalized and the three students generalized the correspondence relationship. A discussion of these students' responses follows.

Two of the students who generalized spontaneously and when prompted (S5 and S8) used algebraic notation to represent their replies. S8's answer to Q1 was: "formula: $(x \times 2) + 6 = 16$; x = number of white tiles". This student used algebraic symbolism "$(x \times 2) + 6$" to express the general relationship. In Q2, Q3 and Q4, this student simply answered the questions, without explaining how he got the result. That was interpreted to mean that the student used the same functional relationship for 8, 10 and 100 tiles, relating the pairs of values $(a, f(a))$ for $a = 8, 10$ and 100 and correctly finding that the number of grey tiles needed would be 22, 26, and 206, respectively.

S6, the third student who generalized spontaneously and when prompted, described the generalization in Q1 in the following words: "they need 16 grey tiles. For every white tile, there are 2 grey tiles, except on the sides, where there are 6. All the whites $x2+6$ on the sides". Hence S6 identified the relationship between variables as well as the constant number (six white tiles on the sides). This student used both verbal and numerical notation to express the relationship. This student's answer to Q5 was: "multiplying the number of white tiles times 2 plus 6 on the sides: $x \cdot 2+6=x$". In other words, S6 used two types of representation: verbal and algebraic, exhibiting a transition from natural to a more general and abstract language.

Note that the three students who generalized, using different representations, expressed the general formula by identifying the correspondence relationship in the function $f(x)=2x+6$.

Only Prompted Generalizations

Sixteen students generalized promptly (when answering Q5). These students expressed the general relationship between the pairs of values (correspondence) verbally. A few representative examples follow.

The students identified the pattern from which they expressed the general formula in a number of ways. In one, eight students described generalization in terms of a rule that in algebraic notation would be represented as $f(x)=2x+6$. The student S14, for instance, answered "you get the answer by multiplying the white tiles times 2 and then adding 6". In this case, as in the other seven, generalization was expressed verbally. Student S3, in turn, replied "multiplying the white tiles by two and adding three at the beginning and three at the end". The pattern detected by this student would be represented in algebraic notation as $f(x)=2x+3+3$. The student S24 adopted a third approach, identifying the pattern to be $f(x)=2(x+2)+2$.

One of these students, S1, used primarily verbal representation, although in conjunction with algebraic symbols. In Q5 the answer was "you need to use $2x$ white tiles +6"; i.e., verbal representation predominated, although with some elements of algebraic symbolism. The implication would seem to be that this student, who used some algebraic symbols sporadically when answering the previous questions, was in route to attaining a more natural and spontaneous use of algebraic symbolism to represent the relationship between variables.

Lastly, the relationship was incorrectly identified by six students in a way that translated to algebraic notation would yield $f(x)=2x+2$. One representative example of this relationship between variables was provided by S9, whose answer to Q5 was "multiply the top and bottom rows by 2 and add 2". Like the other five students, this pupil established a general, albeit mistaken, relationship between the variables.

Discussion and Conclusion

This research supplements other studies focusing on lower grade students' ability to generalize in the context of classroom algebraic functions (e.g., Carraher et al., 2008). Here the emphasis was on generalization by 24 fifth graders who have not received prior instruction on this type of activity, so their answers allow us to explore the generalization in a more spontaneous environment.

For the elementary students, the tiles problem affords the opportunity to explore students' functional thinking, as it enables fifth graders to progress beyond recursive sequences. In fact, they generalized on the grounds of correspondence functional relationships that involved the values of a set of variables.

The overall finding was the existence of two situations in which students generalize: (a) when answering questions about particular instances, and (b) when specifically prompted to generalize. Three students generalized spontaneously, i.e., where the question could be answered without doing so. They consequently used generalization as a strategy to reply to questions involving specific instances. All the students who established a general relationship between the variables (spontaneously or when prompted) based their rules on the correspondence relationship. The students who generalized spontaneously used algebraic notation and verbal representation to express the general relationship between variables. Representation was primarily verbal in students who generalized only when prompted. In line with Blanton et al. (2015), the present authors venture that using algebraic notation would enable students to visualize generalization in fuller detail. That is consistent with the fact that the students who used notation in addition to verbal representation to express relationships did so in questions where generalization was not necessary (spontaneous generalization).

Moreover, the different ways in which students express the functional relationship of correspondence $(f(x)=2x+6; f(x)=2x+3+3; f(x)=2(x+2)+2)$ afforded the opportunity to interpret and understand their thought process when identifying a general relationship between variables.

Lastly, the present findings are related to earlier research results on fifth graders' ability to generalize (Merino et al., 2013), in which verbal representation was also observed to prevail. This study complements ways to analyze the students generalization, considering three aspects: (a) functional relationship, (b) representations, and (c) type of question. The data analysis focused in written students' answers could be complemented with other pieces of evidence (class discussion, for instance) to have more information about how students generalize since some students were able to generalize and, perhaps, did not express it until it was required. Even so, the results support the application of this approach to mathematics teaching in the lower grades, for its favors and enhances algebraic thinking (Blanton et al., 2011).

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