DEVELOPING PURPOSEFUL MATHEMATICAL THINKING: A CURIOUS TALE OF APPLE TREES

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In this paper I explore aspects of the ways in which school mathematics relates to the "real" world, and argue that this relationship is an uneasy one. Through exploring the causes of this unease, I aim to expose some problems in the ways in which context is used within mathematics education, and argue that the use of context does not ensure that the purposes of mathematics are made transparent. I present and discuss a framework for task design that adopts a different perspective on mathematical understanding, and on purposeful mathematical thinking.

Keywords: Context; Purpose; Task design; Understanding

Desarrollo de un pensamiento matemático intencionado: un relato curioso de manzanos

En este artículo exploro aspectos de las maneras en que las matemáticas escolares se relacionan con el mundo "real" y argumento que esta relación es preocupante. Al explorar las causas de esta preocupación, me propongo exponer algunos problemas que surgen de las formas en que se usa el contexto en Educación Matemática y argumento que el uso del contexto no asegura la transparencia de los propósitos de las matemáticas. Presento y discuto un esquema para el diseño de tareas que adopta una perspectiva diferente sobre la comprensión de las matemáticas y el pensamiento matemático intencionado.

Términos clave: Comprensión; Contexto; Diseño de tareas; Propósito

My theme is purpose. I want to approach this theme at, at least, three levels: (a) looking at the intended curriculum, (b) from the perspective of teachers, and (c) through the experiences of children. An over-arching curriculum-level question might be: What is the purpose of teaching mathematics? There are many kinds of possible answers to this question:

- ♦ our economy depends on people with mathematical skills to work in science, technology, engineering, business and economics;

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- ♦ mathematics is a logical discipline which trains the mind;
- ♦ mathematics is an enjoyable activity and part of our cultural heritage; and
- ♦ mathematics is important for understanding the world, and for everyday life.

The last of these is generally foregrounded in curriculum and policy statements. An examination of the aims stated in curriculum documents from a range of countries reveals a fairly consistent message about the importance given to the role of mathematics in enabling learners to relate to the world beyond the classroom: "the need to understand and be able to use mathematics in everyday life and in the workplace has never been greater and will continue to increase" (National Council for Teachers of Mathematics [NCTM], 2000, p. 4); "mathematics and statistics... equip students with effective means for investigating, interpreting, explaining, and making sense of the world in which they live." (Ministry of Education, 2008, p. 26), "mathematics education aims to enable students to… acquire the necessary mathematical concepts and skills for everyday life" (Ministry of Education, 2006, p. 5); "mathematics introduces children to concepts, skills and thinking strategies that are useful in everyday life" (Qualifications and Curriculum Agency, 2008, p. 158); "being mathematically literate enables persons to contribute to and participate with confidence in society" (Department of Education, 2002, p. 4).

I want to explore the implications of this purpose for teaching mathematics, and how the content of school mathematics is shaped by it. I shall focus on the ways in which pedagogic tasks and artifacts are used by teachers in response to this need for everyday relevance. As a starting point, I take a fresh look at the curriculum artifacts that most clearly embody the desire to make school mathematics relevant to everyday life: contextualized word problems.

Despite the high level of agreement within policy level views of the purpose of teaching mathematics, we know that this does not necessarily carry through to the experiences of learners in the classroom. Attempts to identify a core of mathematical knowledge that everyone needs for everyday life are doomed to failure, not least because the needs of everyday life change, both for individuals and for societies. I shall argue that the use of contextualized problems is inherently problematic, and explore some of the reasons why developing purposeful mathematical thinking in the classroom that makes effective connections to everyday life is difficult. Finally, I shall draw on themes from my own research to propose a different perspective of the idea of purpose in school mathematics.

A Curious Tale of Apple Trees

I base my exploration of contextualized word problems on two examples drawn from very different sources. The first is from a textbook published in England in 1887: *The Problematic Arithmetic for the Seven Standards*. The second comes from a very different source: An assessment item taken from the *Programme for International Student Assessment (PISA)* (Organization for Economic Co-operation and Development [OECD], 2006). By serendipity, both problems are set in the context of apple trees.

The 1887 example is typical of a genre that has proved remarkably resilient to change.

> *A gardener gathered 7008 apples from twelve trees and each tree produced the same number. How many from each tree?*

There is a substantial literature within mathematics education, which explores and critiques many aspects of the use and construction of such problems (see, for example, Verschaffel, Greer, & Torbeyns, 2006). I do not wish to engage directly with this literature, but rather to consider two questions in relation to word problems that take a somewhat different perspective from those of previous researchers:

- What purpose did the author have for writing the problem in this way?
- What is the purpose for which the problem is intended to be used in the classroom?

What was the author's purpose? We might suppose that the author chose this context because it appeared a familiar "real" situation, but it is less clear why he—and I assume it was he—did not choose a problem within that context in which the same division calculation could be modeled without attributing obviously unrealistic properties to the apple trees. For example:

> *A gardener gathered 7008 apples and then packed them into twelve boxes with the same number in each. How many in each box?*

Or, with a little less contrivance:

> *A gardener gathered 7008 apples and then packed them into boxes each holding twelve apples. How many boxes?*

It is, of course, impossible to reconstruct the reasons for the author's choices, but it does seem safe to say that a concern with accurately reflecting real life was not the main priority in the composition of this, and other, problems. The choice of an apparently meaningful context of apples and trees is sufficient for the author's main purpose, which is the teaching of mathematics. I shall pick up this point later.

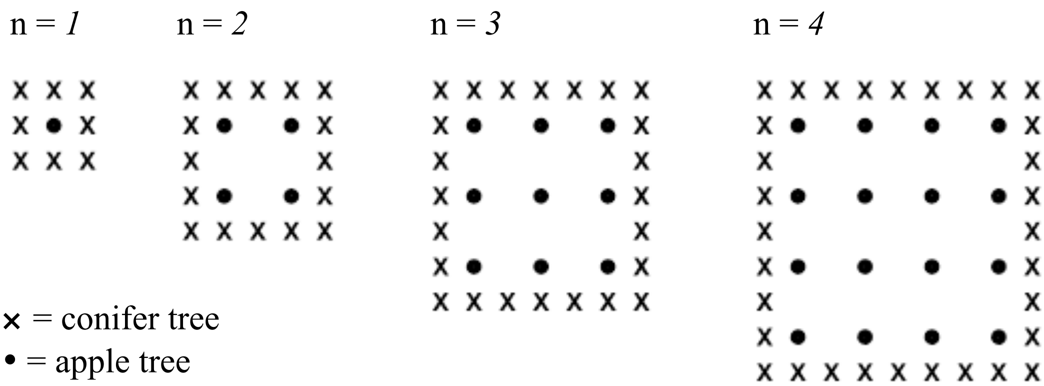
What is the purpose for which the problem is intended to be used in the classroom? We might consider whether it is intended as a teaching resource, to be used to support children in thinking about the process of division, or as part of an assessment to see whether children can apply their knowledge of division to a "real" context. As I shall argue, the purpose for which a problem is intended to be used might offer different perspectives on how we consider its value as a problem.

My second group of apple trees appears in the PISA materials produced by the OECD. PISA is designed to assess mathematical literacy, which is defined as follows.

> *Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.* (OECD, 2003, p. 24)

For the problem I have chosen to focus on, the context is as follows.

> *A farmer plants apple trees in a square pattern. In order to protect the apple trees against the wind he plants conifer trees all around the orchard.*
>
> *Here you see a diagram of this situation where you can see the pattern of apple trees and conifer trees for any number (n) of rows of apple trees:*



Three questions then follow.

Question 1

Complete the table:

n	Number of apple trees	Number of conifer trees
1	1	8
2	4	
3		
4		
5		

Question 2

There are two formulae you can use to calculate the number of apple trees and the number of conifer trees for the pattern described above:

Number of apple trees = n^2
Number of conifer trees = $8n$
where n *is the number of rows of apple trees.*

There is a value of n *for which the number of apple trees equals the number of conifer trees. Find the value of n and show your method of calculating this.*

Question 3

Suppose the farmer wants to make a much larger orchard with many rows of trees. As the farmer makes the orchard bigger, which will increase more quickly: the number of apple trees or the number of conifer trees? Explain how you found your answer. (OECD, 2006, pp. 11-13)

Although the real world context here is more elaborated, it is no less contrived than that in the earlier problem. The idea that the farmer restricts himself to square orchards mirrors the regularity of the trees that magically bear the same number of fruit. The suggestion that he is willing to plant eight conifers to protect a single apple tree stretches credulity in other ways. As part of an assessment of mathematical literacy, it seems an odd choice. I pose the same two questions.

What purpose did the author have for writing the problem in this way? A striking feature of the problem is that, because of the visual presentation, it would work perfectly well if the real world context were removed altogether, and Questions 1-3 were asked simply about the arrays of crosses and circles. The choice of an elaborated "real" context must then relate to the stated aim of PISA to assess capacity to understand the role mathematics plays in the world. Questions 1 and 3 within the problem might, with a little imagination, be seen as something the farmer would need to work out: for any particular size of field, how many of each type of tree will be needed and how would the proportions change. Question 2, however, moves far beyond any realistic use of mathematics. The question may be interesting mathematically, but it is not clear why anyone would need to know the answer.

What is the purpose for which the problem is intended to be used in the classroom? The answer to this is clear: it is part of a written test, intended to assess the "capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics" in ways appropriate to adult life (OECD, 2003, p. 24). As outlined above, I question how effectively the problem achieves that aim, though of course it is only one problem from a larger collection. The problem has not been designed as a teaching resource, to support pupils' understanding of the mathematics involved, although it could be used in that way with a little adaptation.

For example, in a classroom situation, a teacher might ask pupils to find the formulae used in Question 2 for themselves rather than providing them, or extend Question 3 by including rectangular fields with different proportions. This leads to more questions about the relationship between school mathematics and the real world that will illuminate why that relationship is so uneasy.

Why Contextualize School Mathematics?

Clearly one reason for contextualizing school mathematics is to address the curriculum aims quoted earlier: to help children to understand how and why mathematics is used in everyday life, to support their understanding of the world, and to equip them to participate confidently as citizens. There is an implicit expectation that word problems can do this, by acting as "boundary objects" that "inhabit several… worlds… and satisfy the informational requirements of each of them" (Star & Griesemer, 1989, p. 393) and thus bridge between the classroom and everyday life. We have ample evidence that this expectation is not fulfilled. Word problems generally do not function in this way: they fail as boundary objects because they do not satisfy the informational requirements of everyday life.

In a wide ranging study of children's responses to contextualized problems in assessment, Cooper and Dunne (2000) have provided revealing evidence of a difficulty some children, and particularly those from lower socio-economic groups, have in responding correctly. Their findings highlight one of the ways in which word problems fail as boundary objects. They identify a difficulty that is not caused by a lack of understanding of the mathematics, or of the context being used, as has been the case in some other studies. The issue which Cooper and Dunne identify is children's failure to understand the implicit rules of the pedagogic context about how to make use of everyday knowledge. They observed children who approached solving contextualized problems by drawing on aspects of their everyday knowledge in ways that were not intended, such as using the actual price that they have recently paid for a canned drink, rather than using information given in the problem to work it out.

In contrast, children who have learned the rules of the pedagogic game often "perceive school word problems as artificial, routine-based tasks which are unrelated to the real world" (Verschaffel et al., 2006, p. 60) and manage to answer word problems correctly precisely because they largely ignore the context and attend instead to the form of the question. Approaches to "teaching word problems" adopted by teachers, and enshrined in teaching resources, involve strategies for identifying key features of the written problem (the numbers, and words such as altogether which signal the operation to be used) that strip away the specifics of context, which are seen as a distraction. Gerofsky (1996) paraphrases this approach in the following instructions: "I am to ignore… any story elements of this problem, use the math we have just learned to transform… [it]... into correct arithmetic or algebraic form, solve the problem to find one correct answer" (p. 39).

Such strategies prove very effective in terms of operating successfully in the classroom, but clearly work against the espoused curriculum aim of learning to use mathematics to make sense of real world situations. Word problems are ostensibly used to address the curriculum aims of learning the mathematical skills relevant to everyday life, and yet research and established pedagogic practice point to ignoring the context of problems as the best way for children to succeed in solving them.

Further, Verschaffel, Greer, and Torbeyns (2006) discuss a wide range of research, over a long period, which has sought to explore the difficulties that students experience in making the transformation from a "story" format into a mathematical one, and the pedagogic challenges of supporting them to do this. In the face of such evidence of the difficulties that students and teachers experience with word problems, I am led to question why it is that the writers of pedagogic materials and mathematics tests appear as committed as ever to their value in the curriculum.

Some Reasons for the Continued Use of Contextualized Problems

A somewhat cynical answer to the question of the continued popularity of contextualized problems is that contexts are used to "dress up" the mathematics in order to interest and engage learners. The tales of apples trees I have already examined might not appear to be particularly likely to excite school children, but since, as we have seen, the contexts are only laid loosely onto the mathematics, we could easily replace them with something more superficially interesting or relevant: a football crowd being seated in a stadium in which the rows each hold twelve people, a supermarket selling chocolate bars which are delivered in boxes of twelve, the placing of chairs around tables in a school dining room.

I have suggested previously (Ainley, 1997) that there may be two other answers to this question, and exploring them exposes one area of confusion that contributes to the unease in the relationship between school mathematics and the "real" world. The two answers are encapsulated in the traditional format used for school textbooks and resources. This format is, of course, not universal, but it is common enough to reflect, and be reflected in, perceptions about the role of context.

At the beginning, the mathematical topic is introduced and explained. Here it is common for a real world context to be used, partly to provide interest and engagement, but more importantly to support the pupils' understanding of the mathematical ideas. The next section may then contain exercises to practice the ideas and procedures that have been introduced, without the support of contextual examples. Later further contextualized problems are given in which the mathematics that has been practiced is applied to "real" examples. Later still, contextual problems may be used to test whether the mathematical ideas have been properly understood.

Two contradictory ideas are at work here. At first, context is used to support the understanding of a piece of mathematics, by relating it to a familiar situation in the "real" world, and thus offering a model for thinking about the mathematical structures. For example, arranging chairs in rows for a meeting in the school hall might provide a model for multiplication, and a context for the multiplication of relatively large numbers. This is considered to make the introduction of the mathematical ideas easier. This notion is at the heart of the Realistic Mathematics Education (RME) approach: Context is "a characteristic of a task presented to the students: referring either to the words and pictures that help the students to understand the task, or concerning the situation or event in which the task is situated" (Van den Heuvel-Panhuizen, 2005, p. 2).

Later in the text, contextualized problems are used to check that the mathematical ideas have been understood. Being able to solve these problems successfully is an indication that pupils have "really understood" the new mathematical ideas, rather than just being able to perform calculations. The underlying assumption here is that context makes the mathematics harder, and there is evidence, for example from the work of Cooper and Dunne (2000), that in test situations pupils perform better on context-free questions than on contextualized ones. This view is not uncontested. Within the RME approach, carefully designed context in assessment problems is seen as "enhancing the accessibility of problems, contributing to the transparency… of problems, and suggesting solution strategies" (Van den Heuvel-Panhuizen, 2005, p. 2).

A distinction is being made here by Van den Heuvel-Panhuizen (2005) between word problems, in which "the context is not very essential - it can be exchanged for another without substantially altering the problem" (p. 5), and "context problems", in which there is a more intimate connection between the context and the mathematics. I do not find this distinction so clear-cut, perhaps because of my limited experience of RME. However the fact that context can, with careful design, clearly be used to make assessment problems more accessible does not alter the fact that contextualized problems are often used in assessment as being a better way to test "real understanding" than straightforward calculations.

Thus we have two more answers to the question of the purpose for contextualizing school mathematics, and they are contradictory. On the one hand, mathematics is contextualized to make it easier for children to understand, and on the other it is contextualized to make it harder, and test whether they have understood it. Of course this is a somewhat simplistic perspective, since the pedagogic contexts, and particular teacher input, will be very different in teaching situations, where ideas are being introduced, and in testing situations. Nevertheless, I believe that there is a double-think at the heart of our uses of context that is symptomatic of the uneasy relationship between school mathematics and the "real" world.

A Further Confusion

In her plenary lecture at the Seventh Conference of the European Society for Research in Mathematics Education (CERME7), Anna Sierpinska described a difference between "real mathematics" and primary school mathematics as follows (Sierpinska, 2011): (Real) mathematicians model bits of the world in mathematical terms; in primary mathematics we look for aspects of the real world that we can use to model bits of mathematics. This comment is insightful, but, I think, does not quite capture the whole picture. First, this is not only true of primary school mathematics; I think it is true of all of school mathematics. Perhaps more importantly, there is an implicit assumption that these two activities are closely related, if not simply two aspects of the same approach, and that having bits of mathematics modeled in real world contexts will enable children to use mathematics to understand the world.

Returning to national curriculum documents we can find reflections of a two-way relationship. For example, in the English National Curriculum documents there is a claim that: "Mathematics helps children make sense of the numbers, patterns and shapes they see in the world around them" (Qualifications and Curriculum Agency, 2008). In contrast in other equivalent documents—this example is from Singapore—we find claims that opportunities to make connections between mathematics and everyday life "help students make sense of what they learn in mathematics" (Ministry of Education, 2006, p. 8).

The important difference here is one of purpose. When the mathematician uses mathematics to model the world, their purpose is to understand the world better; perhaps to be able to predict earthquakes or stock market crashes, to design efficient traffic systems or bridges. When a teacher chooses a real world context to model a bit of mathematics, their purpose is to teach mathematics; to help children understand the mathematics ideas; or to test that they have understood and can apply them.

I suggest that this is not a trivial difference, and failing to recognize it is potentially serious. When we select real world contexts to model bits of mathematics, we look for simple structures that match those of the mathematical ideas we want to teach or to test. This is what leads us into the magic orchards where trees bear the same numbers of apples, or grow only in rigid square arrays. However, if rather than looking at the two orchards as problems intended to make connections between the classroom and the "real" world, we look them as examples of using the real world to understand mathematics, they seem less problematic. The first does offer a simple model of division. We can imagine the apples distributed across the twelve trees and the large pile of apples once they have been harvested. The second presents a related pair of patterns that can be expressed algebraically.

An Uneasy Relationship

To summarize, I have argued that there are two important, but generally unrecognized, ways in which our thinking about the relationship between school mathematics and the "real" world is muddled. In different ways, they both concern the purposes of the curriculum, and of teachers, for contextualizing mathematics. The first is that we use contextualized problems both to make mathematical ideas easier to understand, and to make them harder, to test that children have understood them. The second is that whilst we intend to use contextualized problems with the purpose of helping children to makes sense of the world, the purpose behind the design of those problems is often to use the world to make sense of the mathematics. If we look at them from this perspective, the continuing widespread use of word problems is both more understandable, and less worrying: they can provide good models for thinking about mathematical ideas. The danger lies in thinking that having used problems in which the world is used to make sense of mathematics, we have also achieved the aim of giving children opportunities to see how mathematics can be used to make sense of the world. The result is that we often achieve the opposite of our intention, leaving children with a view that the context of problems is there to complicate the situation and so best ignored, and a view of mathematics as irrelevant and purposeless.

> *...it was as if there were a kind of check-in desk just outside the classroom door labeled "common sense", and as the pupils filed into the classroom, they left their common sense at the check-in desk saying "Well we won't be needing this in here".* (Wiliam, 1992, p. 3)

Using Mathematics to Make Sense of the World

I now want to shift attention to the ways in which the purposes for learning and doing mathematics may be understood by learners. So far I have suggested that real world contexts are used in school mathematics mainly for the purpose of modeling mathematical ideas, and this is, of course, somewhat simplistic. There are many resources which have been developed to link parts of the mathematics curriculum directly to "everyday" uses of mathematics that are likely to be relevant to children's present or future lives. Whilst this may have some success, there seem to me to be a number of difficulties likely to occur when we try to present the purpose for learning mathematics in this way. What appears to be a logical pedagogic approach may not match the actual, and often more complex, experiences of children outside school. In a study about how children in Hawai'i learn about money, Brenner (1998) illustrated this very strikingly: In school they begin by learning about cents and gradually work up to using dollars, while in their real experience in local shops, dollars are what count, and children often discard cents as useless. This is one reason why children may not recognize the purpose of learning even those aspects of mathematics that might be regarded as most closely related to everyday life, such as measurement, when they learn them in classroom contexts (Ainley, 1991).

Furthermore, in the transition into the classroom real situations get tidied up and simplified, so that they may become the equivalent of the magic orchards. Even situations such as paying tax or buying on hire purchase, which may be concerns in their future adult lives, are unlikely to generate real engagement and interest for children who are still in school.

Research which has drawn on the perspective of situated cognition to explore "street mathematics" may appear to offer a different approach. Lave and Wenger (1991) claim that in out-of-school learning contexts, "learners, as peripheral participants, can develop a view of what the whole enterprise is about" (p. 93). Such opportunities seem to be relatively rare in mathematics classrooms. Whilst I would support Schliemann's (1995) claim that "for meaningful mathematical learning to take place in the classroom, reflection upon mathematical relations must be embedded in meaningful socially relevant situations" (vol. 1, p. 57), the provision of such experiences in the classroom is inherently problematic if the context alone is transferred from the real world, but not a purpose for using mathematics which makes sense to learners.

Creating Opportunities for Purposeful Mathematical Thinking

I have argued so far that the traditional approach of using contextualized word problems, and indeed other more sophisticated approaches which rely on using aspects of the real world to help children to understand specific mathematical ideas, can have only limited success in supporting children to develop the kinds of mathematical thinking that is needed to make sense of the world. I want now to try to address the more difficult question of what else might be required to embed an understanding of mathematics in meaningful socially relevant situations.

I start by looking not at school mathematics, but at "real world" mathematics. We know quite a lot about the mathematical thinking that is developed, even by those with little formal mathematical education, in out-of-school contexts. For example, the methods used are often idiosyncratic, and linked to specific contexts and resources. However, an over-riding feature of "real world" mathematics is that people use it for a clear purpose, to get things done. The people using the mathematics understand why and how it is being used, and it really matters to everyone involved that the answers that are produced are correct.

In my own research, largely in collaboration with Dave Pratt, I have focused on the design of pedagogic tasks that attempt to replicate, in the classroom, this kind of context for mathematical thinking. We have developed a framework for this design, which has two dimensions (Ainley & Pratt, 2002; Ainley, Pratt, & Hansen, 2006). The first dimension focuses on creating tasks that have a clear purpose for learners, within the classroom. Here we are concerned with purpose seen from the perspective of the learner, and the activity that is taking place during the lesson. It is not—necessarily—linked to any specific application outside the classroom, and indeed may be about a situation that is clearly not "real", in the everyday sense. In order to create this sense of purpose for the learner, our tasks typically have an end-product, which might be a real object, such as an efficient paper plane, or a puzzle for other children to solve, or a virtual object, such as a method for scaling dolls' house furniture, or dynamic geometry macros which act as a drawing kit for younger children. In other cases, the end product might be the solution to an intriguing problem, such as understanding the behavior of an unusual die, or finding the height of a giant who has left a particular handprint. In the design of these tasks we also aim to leave flexibility for learners to make real decisions about how they structure their activity, as this supports their engagement and ownership of the outcomes. An important feature of such tasks, which mirrors the use of mathematics in out-of-school contexts, is that the purpose of the task, rather than the teacher, becomes the source of feedback about progress.

We see the design of purposeful tasks as important, but only part of the design challenge. In creating such tasks our purpose, as teachers, is to introduce particular mathematical ideas, and provide opportunities for children to make sense of them. Rather than focusing on the development of procedures—techniques and algorithms, specific rules or formulae—and relationships: links within mathematics, internal structure and consistency, the second dimension of our framework is to build into task design the need for learners to use mathematical ideas in ways that will allow them to recognize what we call their *utility*. By this we mean how and why the mathematics is useful to get things done. Again, from the learner's perspective this does not—necessarily—refers to usefulness in the "real world": Our concern is initially with usefulness within the particular task. The aim of our task design is for children to be able to construct a sense of the kinds of situations in which a particular mathematical idea can be used, and the power that it offers.

We argue that mathematical ideas are complex, and composed of different elements, which here we might categorize as procedures, relationships, and utilities—why, how and when the idea may be useful—. As children construct meaning for a new mathematical idea, connections will be made with existing knowledge, but the pedagogic emphasis placed on the different elements will affect the ways in which those links are made. As I have already argued, traditional attempts to contextualize mathematics generally fail to give real emphasis to utilities. Just as pedagogic approaches that focus mainly, or exclusively, on procedures will result in limited understanding, we suggest that approaches which do not emphasize utilities will also result in impoverished learning in which mathematical knowledge becomes isolated as weak connections are made to existing knowledge of the contexts in which it may be usefully applied.

Some Examples of Purposeful Task Design

The following examples are selected from work over two decades, to give a flavor of ways in which the purpose and utility framework has been applied in a variety of projects covering different areas of mathematics. The first is taken from the primary laptop project in the mid 1990's when Dave Pratt and I introduced the first Mac PowerBooks to a primary school. Amongst the software that we provided was an early dynamic geometry package. The 9- and 10-year-olds enjoyed exploring this to draw pictures, but showed no interest when we tried to introduce geometric construction by challenging them to draw squares that could not be "pulled-apart". We then exploited the partnership the children had with a much younger class in the school to design a task that involved the children in making some drawing tools for their younger partners to use. This involved creating screen objects which could be reproduced many times, and which would retain the same shape when they were dragged to change their position and size. The children engaged enthusiastically with the purpose of making a drawing environment for their younger friends, and as they created sets of shapes—squares, rectangles, different kinds of triangles, roofs, wheels—they experienced the utility of geometrical construction, and of particular geometrical relationships (Pratt & Ainley, 1997).

My next example comes from the Purposeful Algebraic Activity project in which Liz Bills, Kirsty Wilson and I used the purpose and utility framework to explore the use of spreadsheets in the introduction of algebraic notation. We developed a series of tasks for children in the first year of secondary school aiming to give opportunities for them to experience utilities of algebraic notation. The activity I describe here is based on exploring a hundred square by taking a 3 by 3 cross, and finding the total of the numbers on its horizontal and vertical arms. Our focus was on the utility of algebraic notation to show structure (Ainley, Bills, & Wilson, 2005). The first stage of the task was for the children to use formulae to create a "testing cross", highlighted in Figure 1, in which the whole cross was filled in when a number was entered in the central square (see Figure 2). They also used formulae to calculate the vertical and horizontal totals, to support their exploration. As they discovered patterns in their results, they were encouraged to try to say why the totals were always the same.

In the second part of the task children were presented with a story about a teacher who had used this activity for several years with her primary school class, but was bored with the cross shape. The children were challenged to design a new shape that could be used for the same sort of activity; that is, where a numerical pattern would emerge from adding sets of numbers in parts of the shape.

	A	B	C	D	E	F	G	H	I	J
1	1	2	3	4	5	6	7	8	9	10
2	11	12	13	14	15	16	17	18	19	20
3	21	22	23	24	25	26	27	28	29	30
4	31	32	33	34	35	36	37	38	39	40
5	41	42	43	44	45	46	47	48	49	50
6	51	52	53	54	55	56	57	58	59	60
7	61	62	63	64	65	66	67	68	69	70
8	71	72	73	74	75	76	77	78	79	80
9	81	82	83	84	85	86	87	88	89	90
10	91	92	93	94	95	96	97	98	99	100
11										
12										
13		6								
14	15	16	17							
15		26								
16										
17	Column	48								
18	Row	48								

Figure 1. The hundred square and testing cross

	B14 - 10	
B14 - 1		B14 + 1
	B 14 + 10	

Figure 2. Formulae in the testing cross

Using formulae in spreadsheet notations to create the testing cross focussed attention on the structure of the hundred square, which remained the same wherever the cross was placed. In turn this supported the children's articulation of reasons behind the pattern, often based on the symmetry of balancing "plus ten" with "minus ten", and so on. Groups of children used this experience to develop new shapes which also produced interesting relationships when numbers in them were added in particular ways: larger crosses, diagonal crosses, L and H shapes. Their discussion revealed their appreciation of symmetry, and the place-value structure in the square, emphasised by the generalised notation.

Much of my research has been in the area of statistics education, and the comparison between the concerns of mathematics and statistics education is often interesting. In one sense the use of context is much more closely linked to learning and teaching of statistics ideas than is often the case in mathematics. Embedding the teaching of statistical ideas within a problem-solving cycle is a well-established approach (see, for example, Ben-Zvi & Garfield, 2004). The relationship between understanding of statistical ideas, and understanding of context can be complex, as Carlos Monteiro's work has suggested (Monteiro & Ainley, 2004). However concerns about how children perceive the purpose of learning statistical ideas are just as real.

In a project which focussed on supporting children's understanding of graphing, which Dave Pratt and I developed with Elena Nardi, we used the purpose and utility framework in combination with an approach we called *active graphing*, in which tasks required children to make use of graphs as analytic tools during a practical activity, rather than only to present results at the end (Ainley, Nardi, & Pratt, 1998). A series of tasks were designed to encourage a focus on the utility of displaying results graphically, in order to look for patterns, and make decisions about further data to be collected. In their activity in response to these tasks, 8-9 year old children developed increasingly sophisticated ways of talking about their graphs that move seamlessly between references to the graph and to the context, indicating the transparency (Meira, 1998) that the graphs have for them (Ainley, 2001). The example in Figure 3 comes from a task in which children are testing the effects of changing wing length on the time of flight for paper spinners. As Tom describes what he sees in the graph his language appears to encompass both the position of the crosses, and the actions of the spinners. This is one of many examples in which we saw children beginning to gain a sense of the importance of looking beyond individual data points to identify trends (Ainley, Nardi, & Pratt, 2000; Ainley, Pratt, & Nardi, 2001).

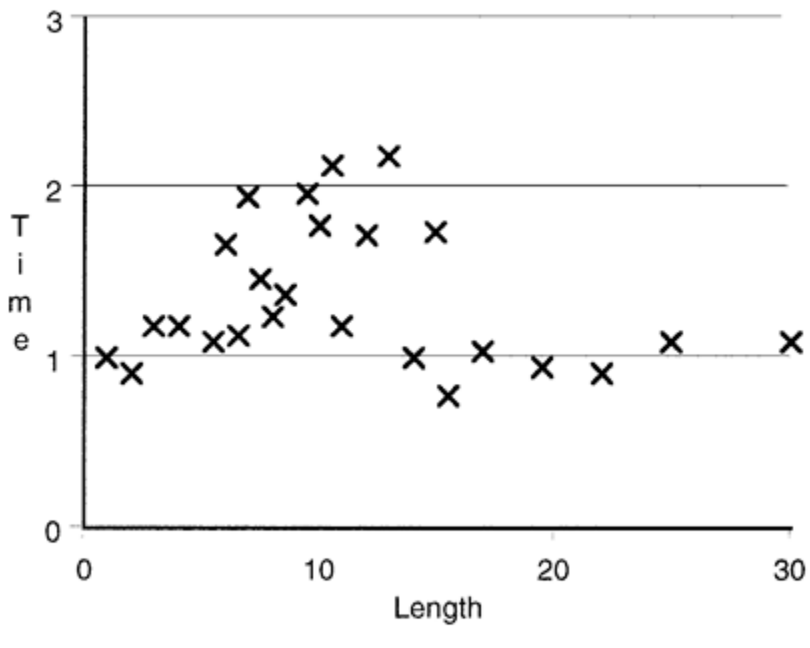


Figure 3. Scatter graph of the flight times for spinner with different wing lengths

Tom's commentaries were as follows:

Tom: I think the [ones] under 10, 15, under 15 are going are the best.

Researcher: Okay, so less than 15 and more than . . . what is this?

Tom: Actually 12 and 9 they are all coming up.

Researcher: Yes, and then what's happening?

Tom: Over 12 they are all going down.

Researcher: They are going down…

Tom: They are just falling instead of spinning.

In some cases, appropriate intervention by the teacher extended children's thinking to include opportunities to consider the utility of repeating experiments and using average values to produce a clearer graph which helped them to identify the optimum wing length (Ainley & Pratt, 2010).

A Perspective From Beyond the Mathematics Classroom

My current research is providing a new perspective on the issue of purpose, at all three levels of curriculum, teachers and learners, by taking me beyond the mathematics classroom in a European cross-curricular project about Inquiry-Based Science and Mathematics Education. In our part of the Fibonacci Project, I am working with my science colleagues, Tina Jarvis and Frankie McKeon, to develop an approach in which the teaching and learning of mathematics and science are integrated through an inquiry approach (Ainley, Jarvis & McKeon, forthcoming). What is emerging from this, apart from rich opportunities for the purposeful use of statistical ideas in the course of scientific experiments, is the explanatory power of mathematics in understanding scientific concepts, and explaining phenomena, within the school curriculum. For example, an understanding of ratio and proportion is important in contexts as diverse as the flight of a paper spinner, why some types of sugar dissolve more quickly than others, the need for a small child to wear more layers of warm clothing than her mother on a cold day, and why elephants have big ears. Working across this curriculum boundary is providing new challenges, and opportunities to extend the framework for task design which focuses on the utility of mathematical ideas by using tasks that are purposeful for children.

Revisiting the Orchard

I began this paper by looking at two examples of attempts to address the purpose for learning mathematics which is expressed on curriculum documents: to enable children to use mathematics to make sense of the world, and to function effectively in it. Through these two tales of apple trees, I have highlighted limitations that they share despite their very different origins, and in particular identified two areas of confusion about the purposes for which contexts are used in school mathematics: to make mathematics easier, or harder, to help explain the world, or to help explain mathematics. In the last part of the paper I have introduced the idea of utility as a third dimension of understanding in mathematics, which can be made available through tasks which are purposeful for children within the classroom. In conclusion, I want to argue that stronger connections between school mathematics and the "real" world can be made by attending to why and how mathematical ideas are powerful, by considering purposes as well as contexts in the mathematics classroom, so that children have an opportunity to see what the whole enterprise of growing apples is about.

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