Mathematical Specialized Knowledge of a Mathematics Teacher Educator for Teaching Divisibility

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The knowledge of Mathematics teachers has been a very prominent focus of attention in the last decades. However, it leaves aside one of the dimensions involved in the development of this type of knowledge, specifically the knowledge of Mathematics teacher educators. In this paper, we discuss a mathematics teacher educator's knowledge in the context of classes on Euclid's division algorithm theorem in a Number Theory course for prospective secondary teachers. Some indicators of this specialized knowledge of mathematics teacher educators are presented and discussed.

Keywords: Euclidean Algorithm; Mathematics Teacher Educator; Mathematics Teachers' Specialized Knowledge; Number Theory

Conocimiento especializado de un formador de profesores de Matemáticas para enseñar divisibilidad

El conocimiento de los profesores de matemáticas ha sido un foco de atención muy activo en las últimas décadas. Sin embargo, tales focos dejan de lado una de las dimensiones involucradas en el desarrollo de dicho conocimiento, específicamente el conocimiento del formador de profesores de matemáticas. En este artículo, discutimos el conocimiento de un formador en el contexto de un curso de Teoría de números, para futuros profesores de secundaria, al probar el teorema del algoritmo de división de Euclides. Se presentan y se discuten algunos indicadores de este conocimiento del formador.

Términos clave: Algoritmo de Euclides; Conocimiento especializado del profesor de matemáticas; Formador de profesores de matemática; Teoría de números

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Mathematics Teachers Educators (MTEs) can be considered as “professionals who work with practicing teachers and/or prospective teachers to develop and improve the teaching of Mathematics” (Jaworski, 2008, p. 1). One of the main roles of MTEs is promoting the development of the knowledge of Prospective Teachers (PTs), in addition to showing them how to establish connections between their training and future practices. Considering that the knowledge of Mathematics Teachers (MT) presents specialized components, as suggested by the perspective of Mathematics Teachers’ Specialized Knowledge–MTSK (Carrillo et al., 2018), MTEs also need to have a special kind of knowledge, which should be related to the specificities of their job of teaching teachers. In this sense, this study intends to contribute to research about the knowledge of MTEs and its role in teacher education, particularly in the context of a Number Theory course for PTs.

Even if the study of Number Theory has potentialities of many connections with school algebra (e.g., integer numbers, divisibility, prime numbers), many MTs understand this topic as being unrelated to their pedagogical practice (Smith, 2002). Divisibility, for example, is frequently treated by PTs as being a trick or a procedure to be memorized, rather than a relation between integer numbers (Zazkis et al., 2003).

The topic of divisibility is present since the earliest years of schooling, including the division of natural numbers, for example. Integer numbers are gradually introduced in the mathematics curriculum, along with some divisibility criteria. In this context, there is a natural underlying question: Why is Euclid’s division algorithm valid? This question is answered in the foundations of the Euclid’s Division Algorithm Theorem (EDAT), frequently presented and proved in Number Theory courses.

The EDAT is a theorem which asserts the existence and the uniqueness of the quotient and the remainder and, as many other mathematical results, it is usually taught and proven by MTEs in the Number Theory courses. The proof is considered a type of mathematical discourse, a kind of narrative that must satisfy the established conventions, and which usually includes text and different visual resources, with the aim of mediating the mathematical ideas involved (Cooper & Zaslavsky, 2017).

In Brazilian universities, as well as many other countries and regions of the world, mathematicians are mostly responsible for the mathematical training of secondary PTs. These professionals “act as teacher educators, without explicitly identifying themselves in this role” (Leikin et al., 2017, p. 2). In this scenario, our focus of research is the knowledge these professionals reveal while teaching. Occasionally being in the role of training PTs, they have solid knowledge on the scientific field of Mathematics, in which they aim to develop studies, acquiring their pedagogical content knowledge with practice (Fiorentini, 2004; Vasco & Climent, 2018) and, similarly to what occur with teachers, based on their previous experiences as students.

This paper is part of a research project which aims to understand and characterize, in the scope of Number Theory, the specialized knowledge of mathematicians who act as Mathematics Teacher Educators. For this article we address the following research question: What elements characterize the mathematical knowledge of a Mathematics teacher educator in relation to Euclid's division algorithm theorem?

LITERATURE REVIEW

Although the knowledge of MTEs, referred to as Mathematics Teacher Educators' Knowledge by Jaworski (2008), is different than both the knowledge of PTs and the knowledge of MTs (Jaworski, 2008; Zopf, 2010; Contreras et al., 2017), it shares common points with them, including knowledge about Mathematics, the pedagogy related to Mathematics, and the curriculum on which Mathematics teachers base their work, whereas the knowledge that is unique to it relates to the literature on the teaching and learning of Mathematics, teaching and learning theories, and research methodologies that investigate teaching and learning in schools/educational systems.

Based on the Mathematical Knowledge for Teaching (MKT) model (Ball et al., 2008), Zopf (2010) observes that the difference between the knowledge of MTEs and the knowledge of MTs lies in the mathematical content. While the latter teach Mathematics, the former teach the knowledge needed to teach Mathematics. The learning goals are also different, since children learn Mathematics for themselves, while teachers learn Mathematics for teaching their students. Therefore, Zopf (2010) proposes the Mathematical Knowledge for Teaching Teachers (MKTT) in order to describe the knowledge of MTEs, which includes the knowledge that is necessary for teaching.

Similarly, Contreras et al. (2017) bases himself on the MTSK model (Carrillo, et al., 2018) to state that the knowledge mobilized by MTEs and teachers have differences when considering Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK), as the knowledge of MTEs is broader in terms of reach and depth, i.e., it has a more coherent and solid theoretical structure; moreover, MTEs have more experience with the validation/construction of mathematical knowledge. On the other hand, PCK relates to the characteristics of learning of PTs, to how the contents should be taught in teacher education, and to the different ways to organize the content of teacher education.

The proposals of Zopf (2010) and Contreras et al. (2017) have common features beyond being grounded in Shulman's model and considering what is already known about the teacher's knowledge to create models pertaining to the knowledge of MTEs. One of these features concerns the teacher education goals: in Zopf's proposal, the teacher education goal is developing the Mathematical Knowledge for Teaching (Ball et al., 2008) of PTs, while in Contreras et al.'s proposal, it is developing their specialized knowledge (Carrillo, et al., 2018). For more information on the development of specialized knowledge in teacher education, see Escudero et al. (2021) and Carrillo et al. (2019).

Furthermore, the models developed by these two research groups are situated in very specific contexts, involving the observation of the practices of experienced MTEs and researchers who already work to develop MKT or MTSK in teacher education. However, when taking into account the peculiarities of MTEs such as the one who participated in our investigation, who are also mathematicians with no prior contact with the MTSK model, instead of trying to base ourselves on the perspective of Contreras et al. (2017), we used the model to search for indicators of a MTE's knowledge, revealed while the participant acted in a teacher education context, focusing on his MK. Thus, our more global aim is also to contribute with the expansion of the MTSK model to the MTE knowledge, also considering the results from Contreras et al. (2017).

Mathematics Teachers' Specialized Knowledge: Theoretical Perspective

In Carrillo et al. (2018), the authors discuss the MTSK model. Teacher's professional knowledge is considered specialized, and the specificities of the mathematical knowledge are considered within three subdomains: the Knowledge of Topics (KoT), the Knowledge of the Structure of Mathematics (KSM) and the Knowledge of Practices in Mathematics (KPM), similarly to PCK, which is also subdivided into three subdomains: the Knowledge of Mathematics Teaching (KMT), the Knowledge of Features of Learning Mathematics (KFLM), and the Knowledge of Mathematics Learning Standards (KMLS). At the center of the model stands the domain of the teacher's beliefs, which are related to all subdomains. Even if we consider the complexity of teachers' knowledge and the intertwined nature of all its dimensions—mathematical and pedagogical, here we focus our attention on the Mathematical dimensions of a mathematician knowledge who acts as Teacher Educator.

Knowledge of Topics

KoT describes what and in what way Mathematics teachers know the topics they teach. It involves thoroughgoing knowledge of mathematical content, such as concepts, facts, rules, and theorems (Carrillo et al., 2018). This subdomain includes the types of problems to which contents can be applied, with their contexts and meanings, properties and principles, procedures and definitions, connections between items pertaining to the same topic, and ways of representing these contents.

It includes four categories related to contents within the definable areas of knowledge making up the Mathematics syllabus: *procedures; definitions,* *properties and foundations; registers of representation* and *phenomenology and applications.*

The *procedures* category refers to teachers' knowledge on *how to do* (e.g., find the solution using algorithms, both conventional and alternative); "when to do" (e.g., sufficient, and necessary conditions to apply an algorithm); why something is done in a certain way (e.g., the principles underlying algorithms), and the characteristics of the resulting object.

The *definitions, properties and foundations* category comprises knowledge of mathematical properties and their underlying principles, in addition to knowledge of mathematical definitions, including how to choose the appropriate sets of properties to characterize mathematical objects. Moreover, the teacher's knowledge of images and examples of mathematical objects also falls within this category.

The *registers of representation* category concerns the knowledge of different ways in which a topic can be represented, e.g., arithmetic and algebraic registers, natural language, graphs, pictographs, and so on.

Finally, the *phenomenology and applications* category is related to the teacher's knowledge of phenomena or situations, organized by topics (Gómez & Cañadas, 2016), also including the teachers' awareness of their uses and applications.

Regarding KoT, in the context of the Number Theory, particularly Euclid's division algorithm theorem, the knowledge of MTEs includes, for example, definitions and results used for proving it, such as the definition of absolute value and the well-ordering principle.

Knowledge of the Structure of Mathematics

KSM describes the teacher's knowledge of connections between mathematical items. There are two types of connections: temporal connections, related to sequencing, associated with the increase in complexity or with simplification; and inter-conceptual connections, related to the demarcation of mathematical objects (Carrillo et al., 2018). It is divided into four categories, as follows:

In the *connections based on increased complexity* category, an item is related to posterior content, and elementary Mathematics is viewed from a more advanced perspective (Klein, 1908). On the other hand, the *connections based on simplification* category acknowledges the links of present with past content; thus, more advanced Mathematics is contextualized in a more elementary content.

The *auxiliary connections* category concerns the necessary participation of an item in major processes. Finally, the *transverse connections* category pertains to contents with common features related by an underlying concept.

In the scope of Euclid's division algorithm theorem, the MTE's KSM includes, for example, connections between it and posterior topics in the Number Theory course, such as linear congruence.

Knowledge of Practices in Mathematics

Practice in Mathematics means that the object of said practice is Mathematics itself. The focus is on the work of doing mathematics, rather than teaching them. It is defined as any mathematical activity carried out systematically, representing a pillar of mathematical creation and conforming to a logical basis for the creation of rules (Carrillo et al., 2018, p. 243). The knowledge of Mathematics teachers about these practices involves proving, justifying, and defining, making deductions and inductions, and giving examples and understanding the role of counterexamples.

KPM can be general or specific to a topic. The former includes knowledge about how Mathematics is developed beyond considering any concept (e.g., knowing the meaning of necessary and sufficient conditions). It relates to the knowledge involved in performing general mathematical tasks, along with knowledge of how a demonstration can be applied, of the different characteristics of definitions (Mamona-Downs & Downs, 2016), of the argumentation practices available (Stylianides et al., 2016), of heuristic approaches to problem solving, and of theory-building practices. On the other hand, the latter relates to a specific instance of general KPM associated with the peculiarities of the topic in question (Carrillo et al., 2018), and concerns, for example, the use of heuristic strategies to address specific topics.

In summary, KPM pertains to the teacher's knowledge about ways of applying, validating, exploring, generating, and communicating mathematical knowledge (Delgado-Rebolledo & Zakaryan, 2019). The KPM of MTEs includes, for example, to know the different types of proof, such as proof by contradiction, used to justify the fact that the remainder is less than the divisor in Euclid's division algorithm theorem, for instance.

CONTEXT AND METHODS

Our investigation had a qualitative approach, with adoption of an instrumental case study (Stake, 2006) as research method to obtain information about the subject's knowledge that could be included in the theory about the knowledge of MTEs. In order to answer the research question, we discuss the moment in which an MTE presents and proves Euclid's division algorithm theorem in a Number Theory course for secondary PTs. The participant Andre (a pseudonym) has a Bachelor's degree, a Master's degree, and a PhD in Mathematics. Since completing the master's degree, his research interests lie in Algebra and Geometry. Andre has been working at the mentioned university for five years, where he teaches students of different undergraduate courses, such as Mathematics, Physics and Chemistry. In the period during which his classes were observed, Andre had been teaching the Number Theory course to undergraduate Mathematics students for the second time in his academic career.

Andre's case study is part of a larger research which aims to understand the knowledge of MTEs, in particular those who teach Number Theory to prospective upper and lower secondary mathematics teachers. The results reported in this paper are exclusively based on this participant.

The Number Theory course in this study is a 15-week course, offered once in each semester as a common discipline for prospective teachers and undergraduate Mathematics students. Furthermore, the PTs are supposed to take these classes in the 6th semester of their undergraduate course. The course includes standard contents of a first course in Number Theory, such as divisibility, prime numbers, linear congruence, Diophantine equations and primitive roots.

Data collection occurred between March and July 2018, in a Brazilian university, comprising interviews, audio recordings and field notes. The first interview was conducted at the beginning of the semester, to clarify points regarding topics that would be taught by the MTE; and a second interview was carried out at the end of the semester, looking for to better understand aspects of the MTE's practice and knowledge that remained unclear, as well as offering feedback on the results of the research.

Class observations and recordings aimed at identifying the evident specialized knowledge in the MTE's practice. There were two types of short interviews: before each class, with the objective of obtaining his previous image of the lesson (Schoenfeld, 2000; Ribeiro et al., 2012), and after each class to discuss aspects associated with the reasons that led Andre to follow a certain way.

In its turn, the researcher's field notes aimed to fully transcribe the content written by the MTE on the blackboard, as well as providing the writing of comments and questions about the events of each class. Starting from the transcript, each class was divided in episodes (Ribeiro et al., 2012), and in this paper we discuss one of such episodes, where Andre proves Euclid’s division algorithm theorem, to present and discuss his knowledge. For clarity purposes, only the transcript of this episode had its lines numbered.

For each episode, the MTE’s mathematical knowledge was identified and then the evidence and the content of that revealed knowledge were organized, structured by the subdomain to which it refers. For the analysis of the Mathematical Knowledge revealed by Andre, we used the categories proposed by Carrillo et al., (2018) assigning an acronym to the indicators (e.g., KSMt1) consisting of the initials of the subdomain in question, plus the representative letter(s) of the associated category, followed by a sequential number according to the order they appear in the text (Table 1). For clarity purposes, this table only includes the categories present in the analysis. To highlight the knowledge revealed by Andre, we inserted a parenthesis with these acronyms accompanied by a brief description of the MTE’s knowledge.

Table 1 presents a synthesis of the subdomains of MK and the categories in which they are divided.

Table 1
Subdomains and categories of Mathematical Knowledge

Subdomains	Categories	Acronyms
KoT	Definitions, properties, and foundations	KoTd1
	Phenomenology and applications	KoTph1
	Procedures	KoTp1
	Registers of representation	KoTr1
KSM	Connections based on simplification	KSMs1
	Connections based on increased complexity	KSMc1
	Transverse connections	KSMt1
KPM	Ways of proceeding	KPMwp1
	Ways of validating	KPMwv1

Note. KoT=knowledge of topics; KSM=knowledge of the structure of mathematics; KPM=knowledge of practices in mathematics.

ANALYSIS AND DISCUSSION

This section will present an overview of the episode selected for analysis in this paper. First, we present a brief interview, conducted before class, in which Andre presents his goals for the day. Then, we present and discuss the episode in which Andre proves EDAT, analyzing his mathematical knowledge. Subsequently, we include the after-class interview, in which the MTE comments on the proof of EDAT.

The interview before class

Researcher: What are your goals for today's class?

Andre: Today's class? Well, I'm going to introduce prime numbers, right? I'm already introducing them with divisibility properties. I can show [the students] that every positive integer is a finite product of primes. Only existence and uniqueness require a little more time, but with existence alone I can already prove that there are an infinite number of primes, and that there are infinitely many primes of the form $4k+3$. If there's time left I intend to introduce the greatest common factor and some properties of the greatest common factor, such as Bezout's theorem... Oh! And Euclid's division algorithm theorem, I'm introducing it today! With proof.

Based on the interview before class, we can note that the MTE intends to present several results in this class, as he does in all classes. This gives a general idea of how his classes are organized, always focusing on presenting as much content as possible. Andre's remark that he plans to present the proof of EDAT refers to the interview held at the beginning of the semester, in which he was asked about the importance of proofs.

EDAT's introduction and its discussion

Andre had introduced the topic of divisibility at the end of the previous class, by presenting its definition and some basic properties. The analyzed episode is part of a class, which the teacher educator started by defining prime numbers. Thereafter he proved the existence part of the Fundamental Theorem of Arithmetic[1], that there are infinite prime numbers, and he defined Greatest Common Factor[2] as well as he proved some properties[3]. In the episode described below, the MTE introduces and proves Euclid's Division Algorithm Theorem (EDAT): considering $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{+}^{*}$, there are unique integers r and q, such that $a = bq + r$, where $0 \leq r < b$.

Andre introduces EDAT (Figure 1) by drawing the students' attention to the connections between it and linear congruence, which will be presented later in the course. Andre also notes that EDAT must be proved in two parts: existence and uniqueness.

Let $a, b \in \mathbb{Z}$, with $b > 0$, so EXIST and are UNIQUE $q, r \in \mathbb{Z}$ such that
$a = b.q + r$ with $0 \leq r < b$.

Figure 1. Euclid's division algorithm theorem written on the blackboard

He starts by considering the set S of all possible non-negative remainders of the division of a by b (Figure 2). Naturally, the first step is to prove that S is not empty, so Andre tells the students to find an integer x such that the expression $a - bx$ is non-negative, which they are unable to do. One of the students apologizes for his incorrect answer, and Andre discusses the importance of the students asking questions as well as the need for observing the details of the theorem's wording, after which he provides the correct answer (Figure 3). Since S is a non-empty set of non-negative integers, it has a minimal element, referred to by Andre as r. Subsequently, the MTE proves that this element satisfies the theorem's conditions (Figure 4), as the existence of r implies the existence of q.

280 (Writing on the blackboard)

281 *Andre*: Now it's time for tonight's main event! Euclid's Division Algorithm Theorem. Let's start by considering two

1 Any integer greater than 1 can be written as a finite product of prime numbers.

2 The Greatest Common Factor of two integer numbers is the largest positive integer number that divides each one of these integers.

3 Such as "If $d = gcd(a, b) \Rightarrow gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$".

282 integers, a and b. Actually, I'm leaving b as positive so there are no problems. So

283 there exist, and are unique, integers q and r, such that a is equal b times q, plus r, with r being positive,

284 but strictly smaller than b. Ok?

Andre's choice of making $b > 0$ [282] means that he is introducing a particular version of EDAT. In the general version, the only condition is that b needs to be a non-zero integer. The MTE probably did this to save time, since considering $b \neq 0$ would divide the proof into more cases. Another possibility is that Andre believes that proving the theorem while keeping b positive would imply the validity of this proof also in case it was negative, although he does not mention this to the students. Regardless of the proof, it would be important to present the full version of EDAT, since this is the one that should be remembered by the students, especially those hoping to become teachers.

285 *Andre*: So, in a few classes taking place

286 sometime in the future, the division algorithm will be a direct

287 consequence of the congruencies we'll find when studying the arithmetic modulo n. But for now, we're

288 proving it with the tools that we have.

In the above transcript, Andre establishes a connection between EDAT and linear congruence [286-287], two different content items with features in common, connected by the underlying notion of divisibility (KSMt1–Knowledge of transverse connections between Euclid's Division Algorithm Theorem and linear congruence). Andre also explicitly makes connections between EDAT and other concepts during an interview at the beginning of the semester:

Andre: [...] for example, I could consider... the division algorithm simply as modular arithmetic. Once we know $\mathbb{Z}_n$, we know that... every number z is congruous with some alpha module n. Right? But what does that mean? That it always exists and is unique. This is Euclid's division algorithm theorem. That's pretty much the same thing, isn't it? So, I can prove Euclid's division algorithm theorem, but when I prove that the $\mathbb{Z}_n$ ring is well defined, that everything works correctly, Euclid's algorithm can be seen as a… consequence. Right? (Transcript of the initial interview).

Andre expressed these connections in the initial interview when he was asked about the topics, he considers fundamental in the Number Theory course. When stating that EDAT can be seen as a consequence of modular arithmetic, he establishes a connection based on increased complexity, viewing EDAT from a more advanced point of view (KSMc1–Knowledge of connections between EDAT and modular arithmetic based on increased complexity). On the other hand, in the same interview, Andre establishes another type of connection:

> Andre: If I'm talking to an elementary school student, I could say: imagine that we have a set of caramels. I want to know if I can make small bags with three caramels each, and then check if every small bag I have has three caramels. This is divisibility. [...] Euclid's algorithm [...] would be understood by the child as the fact that I can take the small bags with three caramels each, and... know exactly how many caramels the spare bag has. (Transcript of the initial interview).

In the above transcript, when asked about an elementary school student's understanding of divisibility and Euclid's division algorithm, Andre contextualizes advanced Mathematics (EDAT) in elementary Mathematics (divisibility) (KSMs1–Knowledge of connections between EDAT and divisibility based on simplification).

When he states that the theorem should be proven in two parts (existence and uniqueness) [289-291], Andre demonstrates that he knows how to prove existence and uniqueness (KPMwp1–Knowledge of how to prove existence and uniqueness by splitting the proof into two parts).

289	*Andre*:	Here [pointing to the blackboard in Figure 1] it says that my proof must
290		be written in two parts. First I need to prove they [q and r] exist, and then I need
291		to prove they are unique. By now, we know that the most
292		difficult part is the existence. Regarding the uniqueness, let us suppose that there exist two, and
293		then see that they are the same.

In addition, Andre knows how to perform the general mathematical task of proving uniqueness [292-293]: assuming there exists two, and then verifying that they are the same (KPMwp2–Knowledge of how to prove uniqueness).

295		(Writing on the blackboard)
296	*Andre*:	First, let's consider this set [Figure 2]. So, let's take all the sets (integers) of the form $a - xb$,
297		where a and b are the numbers I assigned them at the beginning, x is an integer and $a - xb$ is
298		non-negative. Ok? Let's take this subset of integers.

Here, it is possible to identify a heuristic approach to this topic: the choice of an appropriate set S of natural numbers [296-298] to analyze a property of that set, namely, the existence of a minimal element.

Existence Proof

To consider $S = \{a - xb | x \in \mathbb{Z}\ e\ a - xb \geq 0\}$

Figure 2. Set S written on the blackboard

When he starts working with set S, Andre notes that because it is non-empty, since it is a subset of non-negative integers [300-301], it admits a minimal element (KoTd1–Knowledge of the non-empty property of any set of non-negative integers).

299	*Andre*:	I would like to prove that it [S] is not
300		empty. Because it is not just a subset of integers. It is a subset of
301		non-negative integers. This is one of my hypothesis, that these integers are non-negative.
302		We know that a non-empty subset of $\mathbb{Z}_{\geq 0}$ always admits a minimal element. Let's
303		explore this. First, I have to prove that it is not empty. To this
304		end, it is enough to show there is an element in there. Am I right? What is that element?

Then, he reminds the students of the fact that all sets composed of non-negative integers have a minimal element [302], i.e., he refers to the well-ordering principle (KoTd2–Knowledge of the well-ordering principle according to which all sets of non-negative integers have a minimal element). When he refers to the non-negative integers as $\mathbb{Z}_{\geq 0}$ [302], Andre shows that he knows how to represent this subset of integers (KoTr1–Knowledge of how to represent $\mathbb{Z}_{\geq 0}$ for non-negative integers). Additionally, he observes that to prove that S is non-empty, it is necessary to exhibit one of its elements [303-304] (KPMwv1–Knowledge of how to justify that a set is non-empty).

Subsequently, Andre tries to exhibit one of the elements of S, based on this set's characteristics.

319	*Andre*:	No, that is okay. Do not apologize. Do not apologize, this question allows us to see
320		the details, that every detail that is written is important. It's not a and it's not $x = 0$. What is
321		the number that we know is positive? It is b. So, to get around this, I would put the
322		minus in a to obtain a positive sum. The only problem is that I do not know if a is
323		positive or negative. […]
324		(Writing on the blackboard)

325 So, I obtain minus the absolute value of a. Now there's no problem. Since I have the

326 absolute value of a, I can calculate a plus the absolute value of a multiplied by b. b is strictly

327 positive, based on the theorem's initial condition. So, this means that there is at least one.

328 Thus, this value is greater than or equal to zero.

When discussing the importance of the students' questions [319-320], he expresses his belief of the need to perceive and consider all the theorem's conditions, as well as his understanding that this is an aspect to be developed together with his students. By choosing an appropriate x, such that $a - xb \geq 0$, Andre manages to exhibit one of the elements of S [325], showing that this set is not empty [327] (KoTp1–Knowledge of how to demonstrate that S is non-empty).

330 *Andre*: So, S is not empty. If non-empty S is a subset of it ($\mathbb{Z}_{\geq 0}$), S admits a minimal element. If there is a

331 minimal element, I must call it something. The magic is that I call it r.

Here, Andre uses the well-ordering principle (KPMwp3–Knowledge of how to use the well-ordering principle to prove EDAT) to draw attention to the fact that S has a minimal element [330], and names this element r [Figure 3].

$x = -|a| \quad \Rightarrow \quad S \neq \emptyset$ admits minimum.

$a + |a|b \geq 0$ I call it $r \geq 0$

Figure 3. Proof that S is non-empty, written on the blackboard

After naming r, the minimal element of set S [332], Andre tries to justify that it satisfies the conditions of EDAT, i.e., he will verify if $r \geq 0$ and $r < b$ (KPMwv2–Knowledge of how to justify that the minimal element of S meets EDAT's conditions). Thus, Andre notes that if r is an element of S, then $r \geq 0$ [333-334].

332 *Andre*: Then let us see if this r is

333 exactly what I want. Ok! First property: r, which I already know is

334 greater than or equal to zero. Since r is an element of this form, the minimal element of this set [set S],

335 is an element of it. All elements in this set are non-negative, including r.

In the next excerpt, Andre reveals that he knows how to construct set S (KPMwp4–Knowledge of how to construct the necessary set S of non-negative integers that is in the core of the proof of EDAT, satisfying the theorem's conditions) such that its minimal element (r) satisfies the conditions of EDAT ($a = bq + r$) [337-340].

337	*Andre*:	What is the set that I considered? I chose exactly all integers
338		of the form $a - xb$. This means that I'm considering all relations
339		in which a is equal to x times b plus one integer. Thus, set S is chosen
340		precisely to satisfy this relationship. Am I right? So, I need to define […]
341		the smallest of the remainders, and then check if it satisfies
342		this condition [$a = bq + r$, in the theorem].

Continuing the proof, Andre intends to prove, by contradiction [380-382], that the remainder is less than b.

379		(Writing on the blackboard)
380	*Andre*:	I need to prove that r is strictly less than b. Let's
381		prove it by contradiction. So, […] I will assume that r is greater than or equal to b. It's the
382		opposite. Thus, this will result in a contradiction.

Figure 4 shows what Andre wrote on the blackboard about this.

I want to prove $r < b$.

By contradiction, I will assume that $r \geq b$.

Then, $0 \leq r - b = a - bq - b = a - (q+1)b$.

I name $q = x$ such that $a - bq = r$

$$\Rightarrow\ r - b \in S$$

$$r - b < r.$$

Figure 4. Proof that r is strictly less than b

Now, he explains that to do this, it is necessary to deny the thesis [400-401] (KPMwv3–Knowledge of the steps of a proof by contradiction). Then, after assuming that $r \geq b$, Andre does some algebraic manipulations and considers $q = x$, leading to the conclusion that $r - b$ belongs to S, which contradicts the fact that r is the minimal element in this set. This kind of proof by contradiction is frequently used in Algebra. When the thesis is contested, a conflict in relation to the minimality of an element arises (KPMwv4–Knowledge of how to prove that the remainder is less than the divisor based on a conflict in relation to the minimality of an element of set S).

400	*Andre*:	Finding a contradiction means that the

401		hypothesis I'm basing myself on is absurd. So, it is impossible for r to be greater than or equal
402		to b, which implies that r must be strictly smaller than b. In this way, we
403		(…) prove that there are those integers
404		q and r satisfy the initial condition.

Once the existence of the quotient and the remainder has been proven, Andre begins proving the uniqueness of these elements by assuming that there are a pair of quotients and a pair of non-negative remainders that are smaller than the divisor, both satisfying the theorem's decomposition (Figure 5).

Uniqueness Proof

Suppose that we have q, q', r, r' such that

$$a = q'b + r'$$
$$a = qb + r$$

Figure 5. Two decompositions written on the blackboard.

Thus, he concludes that each of these pairs are equal, i.e., based on the supposition that there are two Euclidean decompositions, he was able to prove that these decompositions coincide. Both decompositions must result in the same dividend, so by conveniently rearranging this equation, Andre concludes that the divisor must divide the difference of the remainders (Figure 6).

$$0 = a - a = (q - q')b + r - r'$$
$$b|r' - r$$

Figure 6. Conclusion written on the blackboard

Then, since the difference of the remainders is smaller than the divisor, the conclusion is that the only possibility for this difference is zero, which implies that the remainders are equal.

405		(Writing on the blackboard)
406	*Andre*:	I've found these two numbers, now I have to prove that both
407		are unique. And to prove that, like I said earlier, let's assume
408		there are two. What does that mean? That there are q, q' and r, r' integers such that a can be expressed
409		as $q'b + r'$ or as $qb + r$. So we have two representations of a in the form we want.

This equality and the fact that the divisor is positive allows inferring that the quotients' difference is also zero; thus, the quotients are equal (Figure 7).

$$0 = a - a = (q - q')b + r - r'$$

$$b | r' - r \qquad \begin{cases} r - r' < b \\ r' - r < b \end{cases} \Rightarrow |r - r'| < |b| = b$$

$$\Downarrow$$

$$r = r'$$

$$\Rightarrow q = q'.$$

Figure 7. Conclusion written on the blackboard

By assuming that there are a pair of quotients (q, q') and a pair of non-negative remainders (r, r') that are smaller than the divisor, both satisfying the theorem's decomposition (Figure 5), Andre evidences that he knows how to prove the uniqueness of these elements [406-409] (KPMwp5–Knowledge of how to prove the uniqueness of the remainder and the quotient in the proof of EDAT).

412 (Writing on the blackboard)

413 *Andre*: So, let's consider that I can write zero as $a - a$. There's

414 nothing complicated about that, right? But now the same thing

415 has two different descriptions. So, this equality is $q - q'b$ plus $r - r'$. First

416 I know that b divides $r' - r$. It's because if it is equal to zero, I can put $r' - r$ on the other

417 side, so if $r' - r$ is equal $q'b$, that means b is a divisor of both of them.

After some algebraic manipulations and using the concept of divisibility [413-415] (KoTd3–Knowledge of the fact that, according to the definition of divisible, an integer a divides an integer b if b is a multiple of a), Andre states that the divisor must divide the difference of the remainders [415-417] (Figure 6). Subsequently, he claims that the absolute value of this difference is less than b [418].

418 *Andre*: Next, I know that $r - r'$ is smaller than b. That's explained by the hypotheses. (…)

419 This proof will differ a bit from the proof

420 in the textbook, because I condensed it a little. In the textbook, there's another one that is

421 equivalent to it.

Here, Andre mentions that he opted for a different proof than the one described in textbook [418-421], showing his preference for a shorter, or, as the MTE himself observes, more condensed version of the proof of uniqueness, which students may find a little harder to understand.

425		(Writing on the blackboard)
426	*Andre*:	So, for the same reason I told you
427		that $r - r'$ is smaller than b, $r' - r$ is also smaller than b. […]
429		But when we have these two differences, the absolute value of $r - r'$ is
430		less than the absolute value of b, which in this case is b because I'm assuming it's
431		positive. Right? Ok. But since it's the absolute value, it implies that $r = r'$.

Considering the first and second remarks, Andre concludes that b divides the difference of the remainders and, at the same time, the difference of the remainders is less than b. Thus, this difference has to be zero, so $r = r'$.

440		(Writing on the blackboard)
441	*Andre*:	And if $r = r'$, then $(q - q')b = 0$, but if b is strictly positive, then
442		q must be equal to q'.

Based on the fact that $r = r'$ and that b is positive, Andre concludes that $q = q'$ [441-442], thus successfully proving EDAT (Figure 7).

We can note that this way of proving the theorem is very succinct (especially in relation to uniqueness), demonstrating the MTE's beliefs about elegant proofs. Andre was asked about proofs in the after-class interview and clarified his views on this matter, as can be seen in the next section.

After proving EDAT, Andre observes that this result may be applied, for example, to the proof of infinitude of prime numbers of the form $4k + 3$, which reveals his awareness of the uses and applications of EDAT (KoTph1–Knowledge of the fact that Euclid's division algorithm theorem has applications in Number Theory, such as in the proof of infinitude of prime numbers of the form $4k + 3$). The table 2 summarizes the mathematical knowledge revealed by Andre during the class while presenting and proving Euclid's division algorithm theorem.

Table 2
Subdomains, categories, and indicators of Andre's Mathematical Knowledge

Categories	Knowledge indicators
	KoT
Definitions, properties and foundations	KoTd1 –Knowledge of the non-empty property of any set of non-negative integers
	KoTd2 –Knowledge of the well-ordering principle, according to which all sets of non-negative integers have a minimal element
	KoTd3 –Knowledge of the fact that, according to the definition of divisible, an integer a divides an integer b if b is a multiple of a
Phenomenology and applications	KoTph1 –Knowledge of the fact that Euclid's division algorithm theorem has applications in Number Theory, such as in the proof of infinitude of prime numbers of the form $4k + 3$
Procedures	KoTp1 –Knowledge of how to demonstrate that S is non-empty by indicating one of its elements
Registers of representation	KoTr1 –Knowledge of how to represent non-negative integers for $\mathbb{Z}_{\geq 0}$
	KSM
Connections based on simplification	KSMs1 –Knowledge of connections between EDAT and divisibility based on simplification
Connections based on increased complexity	KSMc1 –Knowledge of connections between EDAT and modular arithmetic based on increased complexity, considering EDAT as a consequence of the modular arithmetic
Transverse connections	KSMt1 –Knowledge of transverse connections between Euclid's Division Algorithm Theorem and linear congruence, connected by the underlying notion of divisibility
	KPM
Ways of proceeding	KPMwp1 –Knowledge of how to prove existence and uniqueness by splitting the proof into two parts
	KPMwp2 –Knowledge of how to prove uniqueness by assuming that there are two elements and then verifying if they are the same
	KPMwp3 –Knowledge of how to use the well-ordering principle to prove EDAT
	KPMwp4 –Knowledge of how to construct the necessary set S of non-negative integers that is in the core of the proof of EDAT, satisfying the theorem's conditions

Categories	Knowledge indicators
	KPMwp5 –Knowledge of how to prove the uniqueness of the remainder and the quotient in the proof of EDAT
Ways of validating	KPMwv1 –Knowledge of how to justify that an arbitrary set is non-empty by presenting one of its elements
	KPMwv2 –Knowledge of how to justify that the minimal element of S meets EDAT's conditions, i.e., $r \geq 0$ and $r < b$
	KPMwv3 –Knowledge of the steps of a proof by contradiction: denying the thesis and then confirming it by finding a contradiction
	KPMwv4–Knowledge of how to prove that the remainder is less than the divisor based on a conflict in relation to the minimality of an element of set S

Note. KoT= Knowledge of Topics; KSM= Knowledge of the Structure of Mathematics; KPM=Knowledge of Practices in Mathematics.

The after-class interview

Researcher: About Euclid's division algorithm theorem...

Andre: Yes.

Researcher: You commented that you simplified the proof of the textbook.

Andre: Yes!

Researcher: Do you remember what else was there?

Andre: It is this part of $qr' - r < b$, the $r - r' < b$, noting that the rest has to be between 0 and b.

Researcher: Did you think it was not necessary?

Andre: No, it's another proof, equivalent, mine is shorter. So it's more elegant.

Researcher: I understand.

Andre: There is not a single proof. But usually, the shortest proofs are the most elegant. And with these two observations, without having to write an extra part, I already got the result I wanted.

According to Lai and Weber (2014), notwithstanding mathematicians are in general responsible for the teaching of advanced university mathematics courses, their training focuses on writing proofs for disciplinary, rather than pedagogical, purposes. In this case, the disciplinary purposes are more related to proofs produced to advance the discipline, rather than to proofs produced to facilitate understanding of the concepts involved. In fact, Andre seems more interested to conclude the uniqueness of the proof to move on to the next concepts in the course.

On the other hand, even when aiming to present pedagogical proofs, Weber (2012) and Harel and Sowder (2009) found that mathematicians reported have a limited pedagogical arsenal with which to achieve their pedagogical goals with respect to proof. These limitations are naturals and understandable, since many mathematicians do not receive any pedagogical preparation (Fiorentini, 2004), which is not an obstacle to teaching their courses. Although it is not an obstacle, this lack of pedagogical arsenal can cause disadvantages in teacher education, since the MTE aims and practices relate to teach mathematics to prospective teachers without making the connections to the future mathematical practices – focus on the mathematics instead of on the knowledge to teach mathematics – (Zopf, 2010).

The appreciation for elegant proofs (Alsina & Nelsen, 2010) is common among mathematicians, although such proofs can represent obstacles for students, in particular for prospective mathematics teachers, who need to develop specialized knowledge, concatenating mathematical and pedagogical knowledge, to perform their profession. Considering that "[…] a proof is an argument to convince the reader that a mathematical statement must be true" (ibid, p. xix), one can consider that the reader cannot be convinced of this truth if it does not fully understand the proof.

SOME CONCLUSIONS AND FINAL COMMENTS

According to Lesseig (2016), studies documenting the lack of understanding of teachers about proofs suggest that the prominence of their role should be emphasized in teacher education. One way of understanding what mathematical knowledge is required to support the use of proofs to teach Mathematics in schools is to investigate the mathematical knowledge of MTEs.

In this paper, we analyzed the mathematical knowledge of a mathematician teaching a course for prospective teachers, based on the perspective of MTSK. To this end, we searched for evidence of the MTE's knowledge of topics (definitions, properties, procedures, registers of representation, and phenomenology and applications in the topic divisibility), knowledge of the structure of mathematics (connections based on increased complexity, connections based on simplification and transverse connections involving EDAT), and knowledge of practices in mathematics (ways of proceeding and ways of validation).

Considering that mathematical argumentation, reasoning, justification, and proof indisputably constitute an important field of mathematical competencies (Bersch, 2019; Alfaro et al., 2020), we obtained indicators of Andre's KPM related to ways of proceeding and ways of validation. These indicators also contribute to the investigation of the KPM of the mathematics teachers at the university level (e.g., Vasco & Climent, 2018; Delgado-Rebolledo & Zakaryan, 2019).

The focus is not to evaluate or to prescribe which should be the knowledge of MTEs. Our interest was investigating the knowledge of the MTE who participated in our case study, considering the Brazilian teacher education context, where mathematicians are responsible for teaching future teachers. In this sense, our findings can support the development of a model for the specialized knowledge of MTEs, as suggested by Contreras et al. (2017).

Analyzing Andre's teaching practices and looking for indicators allowed us to identify categories of this MTE's mathematical knowledge. Since Number Theory, which naturally includes the topic of divisibility, is still a course where both prospective and experienced teachers reveal difficulties (e.g., Smith, 2002; Zazkis et al., 2003), it requires further research, focusing on the articulated discussions between the mathematical and pedagogical knowledge revealed by MTEs when teaching divisibility.

Another pertinent question is whether there are differences in the mathematical and pedagogical knowledge revealed by MTEs with dissimilar profiles and experiences. Furthermore, we suggest that future studies further investigate the specialized knowledge of mathematicians working in teacher education in a context of partnership with mathematical educators.

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